LEARN

Imagine you drop a video online. It goes viral, and the views grow continuously at a massive rate. You want to know exactly how long it will take to hit 1 million views so you can launch a merchandise drop. Your unknown variable is *time*, and it is trapped in the exponent! To model this real-world situation and rescue that variable, you need the ultimate algebraic tool: the logarithm.

Expand the section below to explore the continuous exponential model.

The Continuous Model +

For situations that grow or decay continuously (like viral metrics, bacteria, or financial investments), use the continuous exponential model:

$$A = Pe^{rt}$$

- A: The final **A**mount you want to reach.
- P: The **P**rincipal, your starting amount.
- r: The growth **r**ate (written as a decimal).
- t: **T**ime
- e: A special mathematical constant (≈ 2.718); you'll explore exactly where it comes from on the next page.

Where Does e Come From?

The number $e \approx 2.71828...$ is one of the most important constants in mathematics, but where does it come from?

The Key Question: What happens if you compound interest more and more frequently?

Start with **1 dollar** at 100% annual interest. Observe what happens as you add more compounding periods per year:

- Compound **once** (annually): $\left(1+\frac{1}{1}\right)^1 = 2.000$
- Compound **10 times**: $\left(1+\frac{1}{10}\right)^{10} \approx 2.594$
- Compound **100 times**: $\left(1+\frac{1}{100}\right)^{100} \approx 2.705$
- Compound **1,000 times**: $\left(1+\frac{1}{1{,}000}\right)^{1{,}000} \approx 2.717$
- Compound **infinitely** (every instant): $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.71828$

The takeaway: e is not an arbitrary number; it is the value whose growth naturally settles when compounding happens at every possible instant. That is exactly why it appears in $A = Pe^{rt}$—the model assumes growth is happening continuously, all the time.

Think about it: Why doesn't $\left(1+\frac{1}{n}\right)^n$ keep growing to infinity as n gets larger?

Reveal the answer +

As n grows, the base $\left(1+\frac{1}{n}\right)$ gets closer and closer to 1, so each additional compounding period adds a smaller and smaller boost to the total. Meanwhile, the exponent n is growing, but it can't keep up because the base is shrinking so fast.

These two effects (a growing exponent and a shrinking base) are in a tug-of-war, and they balance each other out perfectly. The result doesn't shoot to infinity; instead, it settles at $e \approx 2.718$.

This is the idea of **diminishing returns**—the more frequently you compound, the less each additional period actually contributes.

Why the Natural Logarithm ($\ln$)?

It is said "use $\ln$ because the base is e," but **why** does that work? Let's justify it.

Notation Check: $\ln$ is just shorthand for $\log_e$.

These two expressions mean exactly the same thing:

$\ln(x) = \log_e(x)$

The subscript e is dropped because the natural logarithm appears so frequently in science, economics, and mathematics that it has earned its own symbol. Any time you come across $\ln$, mentally replace it with "log base e"; the rules work identically.

Why base e? Because the exponential uses e as its base, and the only logarithm that cancels e cleanly is the one whose base is also e.

Expand each section below to see exactly why $\ln$ is the right tool for the job.

The Inverse Principle: Matching Bases +

Every exponential function b^x has an inverse—the logarithm with the **same base—**written $\log_b$.

By definition, $\log_b(b^x) = x$.

The logarithm and the exponential cancel each other because they are inverse operations, like multiplying and dividing. This holds for **any** base $b > 0$, $b \neq 1$.

Why Not $\log_{10}$ or $\log_2$? +

The exponential uses base e, so you need the logarithm whose base is also e; that's exactly what $\ln$ is.

$\ln(x) = \log_e(x)$

Applying $\log_{10}$ to $e^{0.15t}$ would **not** simplify cleanly: you'd have $\log_{10}(e^{0.15t}) = 0.15t \cdot \log_{10}(e)$, which leaves an extra constant and complicates the algebra. $\ln$ eliminates that extra step because $\log_e(e) = 1$.

Formal Verification +

You can verify the inverse relationship step by step:

Step 1: By definition, $\ln(e) = \log_e(e) = 1$ (since $e^1 = e$).

Step 2: Apply the logarithm power rule: $\ln(e^{0.15t}) = 0.15t \cdot \ln(e)$

Step 3: Substitute $\ln(e) = 1$: $0.15t \cdot 1 = 0.15t$

The $\ln$ and e don't magically disappear; they reduce to 1 because of this inverse relationship, leaving only the exponent.

If a different equation used base 2, for example, $200 = 2^{5t}$, which logarithm should you apply to both sides to isolate t?

Select one

1. The natural logarithm: $\ln$

   The matching principle applies here: the base of the logarithm should equal the base of the exponential. Consider which base this equation uses.

2. The common logarithm: $\log_{10}$

   $\log_{10}$ pairs with base 10. Consider which base this equation uses and select the logarithm that matches it.

3. The base-2 logarithm: $\log_2$

   Since $\log_2(2^x) = x$, applying $\log_2$ to both sides isolates the exponent. Match the log's base to the exponential's base every time.

4. Any logarithm works equally well

   Any logarithm can technically work using the Change of Base formula. The most efficient approach is choosing a logarithm whose base matches the exponential's base directly. Which base does this equation use?

Expand the section below to discover the four steps you'll use to solve any exponential equation.

The 4-Step Solving Process +

No matter the scenario, these four steps will get you to the answer:

1. **Model:** Set up your base equation.
2. **Isolate:** Get the exponential base totally alone.
3. **Logarithm:** Match the logarithm's base to your exponential base! Use $\ln$ to cancel base e, use $\log$ for base 10, and use $\log_2$ to cancel base 2.
4. **Evaluate:** Divide and use technology to find the real-world answer.

Example 1: Video Views

Solving for Time (t)

Remember the viral video from the start of the lesson? The one you dropped online, experienced it exploding, and needed to time your merchandise launch. Let's put everything you've learned to work on it.

Your video starts with **5,000 views**. It grows continuously at a rate of **15% per hour**. You want to know exactly how long it will take to hit **1,000,000 views**.

Select each tab below to review the variables and set up the equation.

Variables | The Equation

P: The Principal (or initial) starting amount.

e: Euler's number (≈ 2.718), which is the base of all continuous growth. It emerges from compounding infinitely often, making it the natural choice for modeling real-world growth that never stops.

r: The rate (as a decimal).

t: The time.

Variables | **The Equation**

Let's plug your numbers into the model!

$A = 1{,}000{,}000$

$P = 5{,}000$

$r = 0.15$

Equation: $1{,}000{,}000 = 5{,}000e^{0.15t}$

Expand each step below to walk through the full solution.

Step 1. Model: Set up the equation +

Identify your known values and substitute them into $A = Pe^{rt}$.

- $A = 1{,}000{,}000$ (the target view count)
- $P = 5{,}000$ (the starting views)
- $r = 0.15$ (15% growth rate as a decimal)
- $t = ?$ (what you're solving for)

Equation: $1{,}000{,}000 = 5{,}000e^{0.15t}$

Step 2. Isolate: Get the exponential alone +

Divide both sides by the initial value, which is 5,000:

$$\frac{1{,}000{,}000}{5{,}000} = e^{0.15t}$$

$$200 = e^{0.15t}$$

The exponential is now isolated. The video needs to reach 200 times its starting size.

Step 3. Logarithm: Match the base +

The base is e, so apply the natural logarithm ($\ln$) to both sides:

$$\ln(200) = \ln(e^{0.15t})$$

Because $\ln$ and e are inverse operations, they cancel on the right side:

$$\ln(200) = 0.15t$$

Step 4. Evaluate: Solve and interpret +

Isolate t by dividing both sides by 0.15:

$$t = \frac{\ln(200)}{0.15}$$

Use your calculator's $\ln$ button:

$t \approx 35.32$ **hours**

The video will hit 1,000,000 views in about 35 hours.

THINK

Example 2: Financial Investment

Solving for Rate (r)

The Scenario: A client wants to triple their investment from $8,000 to $24,000 in exactly **10 years**. What continuous interest rate do they need to make it happen?

Step 1. Model: Your client needs $24,000 from a starting $8,000 in exactly $t = 10$ years. Which equation correctly models this situation?

Select one

1. $24{,}000 = 8{,}000e^{10r}$

   $A = 24{,}000$, $P = 8{,}000$, $t = 10$ are all known — r is the unknown to solve for.

2. $24{,}000 = 8{,}000e^{0.07 \cdot 10}$

   r is what we're solving for, so it should remain as an unknown in the exponent rather than a specific value.

3. $8{,}000 = 24{,}000e^{10r}$

   Take another look at which value is the starting point P and which is the final goal A — think about where the client begins and where they want to end up.

4. $24{,}000 = 8{,}000 \times 10r$

   This formula models simple interest. For continuous compounding, use the exponential model $A = Pe^{rt}$.

Step 2. Isolate the Base: Divide both sides of $24{,}000 = 8{,}000e^{10r}$ by 8,000. What is the result?

Select one

1. $3 = e^{10r}$

   $24{,}000 \div 8{,}000 = 3$. The exponential is now isolated.

2. $16{,}000 = e^{10r}$

   Try dividing both sides by the starting amount P rather than subtracting — division is what isolates the exponential.

3. $24{,}000 = e^{10r}$

   Divide both sides by the principal P to fully isolate e^{10r}.

4. $3{,}000 = e^{10r}$

   Division is the right operation here. Verify the calculation uses the complete value of P as the divisor.

Step 3. Match the Base: Apply the natural logarithm to both sides of $3 = e^{10r}$. Which is the correct result?

Select one

1. $\ln(3) = 10r$

   Because $\ln$ and e are inverse operations, $\ln(e^{10r}) = 10r$. The exponent drops down.

2. $\log(3) = 10r$

   $\log_{10}$ pairs with base 10. Since the exponential uses base e, select the logarithm whose base is also e.

3. $\ln(10r) = 3$

   Apply $\ln$ to the isolated exponential result from Step 2, not to the exponent itself.

4. $e^3 = 10r$

   $\ln$ is an inverse operation that brings the exponent down. Consider what $\ln(e^{10r})$ simplifies to.

Step 4. Evaluate: Divide both sides of $\ln(3) = 10r$ by 10 to isolate r. What is the required interest rate, and what does it mean for the client?

Select one

1. $r = \dfrac{\ln(3)}{10} \approx 11\%$: the client needs at least 11% continuous interest to triple their money in 10 years.

   Dividing $\ln(3)$ by 10 isolates $r \approx 11\%$ — that is the continuous rate the client needs.

2. $r \approx 15.69\%$: this is the time it takes at 7%, not the required rate.

   15.69 represents a duration in years. Return to $\ln(3) = 10r$ and determine what operation isolates r from its coefficient.

3. $r \approx 1.099\%$: this is $\ln(3)$ without dividing by 10.

   $\ln(3)$ equals the product $10r$, not r alone. One more step will isolate r.

4. $r \approx 6.93\%$: this uses $t = 20$ instead of $t = 10$.

   Review the value of t stated in the scenario and use that as the divisor.

The Same Steps, Different Unknowns

You've now solved $A = Pe^{rt}$ twice—once to find **how long** growth takes, and once to find **what rate** is required. Look at the two algebraic paths side by side. The steps are identical—only the unknown changes.

Video Views: Solve for t

What you know: A, P, r

What you need: t

Step 1: Model

$$1{,}000{,}000 = 5{,}000e^{0.15t}$$

Step 2: Isolate the Base

$$200 = e^{0.15t}$$

Step 3: Match the Base

$$\ln(200) = 0.15t$$

Step 4: Evaluate

$$t = \frac{\ln(200)}{0.15} \approx 35.32 \text{ hours}$$

Financial Planner: Solve for r

What you know: A, P, t

What you need: r

Step 1: Model

$$24{,}000 = 8{,}000e^{10r}$$

Step 2: Isolate the Base

$$3 = e^{10r}$$

Step 3: Match the Base

$$\ln(3) = 10r$$

Step 4: Evaluate

$$r = \frac{\ln(3)}{10} \approx 11\%$$